CGL 9. Poisson Distribution.

Another discrete distribution is called the Poisson Distribution. It is sometimes referred to as the rare event distribution. The Poisson Distribution is used to calculate the probability of any number of some event when the average number of the events is known for independent time and/or spatial periods.

For the periods to be independent the probability distribution for the occurrence of the event would be the same over the time and/or spatial units.

The Poisson variable is defined as

X = number of occurrences in independent time and/or spatial periods or units

The values of a Poisson variable are $x = 0, 1, 2, \ldots$. The values of a Poisson variable are always 0, 1, 2, and so forth on all integers to positive infinity.

The parameter of the Poisson distribution is the average number of the occurrence for the time and/or spatial unit. The traditional symbol for this parameter is the lower case Greek letter for l, which is lamda, λ. Some text books designate this parameter as μ, since it is the mean.

If a variable, X, has a Poisson distribution with parameter λ, the probability distribution is described as $X \sim Poi(\lambda)$. This is read as, "the variable X is distributed according to the Poisson distribution with parameter lamda.

The Probability Mass Function for the Poisson random variable is the following:

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad \text{where } x = 0, 1, 2, \ldots$$

The expected value or mean of the Poisson random varible is λ and the variance is also λ. This is one of the few probability distributions where the mean and variance are always equal.

In symbols,

$EX = \mu = \sum x \cdot p(x) = \lambda$. Notice the mean could be calculated as the sum of each x value multiplied by its probability. This would be an infinite sum for the Poisson variable and reduces to the value λ.

$$\text{Variance of } X = \sigma^2 = \sum x^2 \cdot p(x) - \mu^2 = \lambda$$

$$\text{Standard Deviation of } X = \sigma = \sqrt[+]{\sigma^2} = \sqrt[+]{\lambda}$$

Poisson Example.

Imagine that the average or mean number of DUI's in a certain college town per weekend night is 2.2. If the weekend nights are assumed independent the the number of DUI's could be considered as a Poisson random variable with parameter $\lambda = 2.2$.

The variable is

X = number of DUI's on a weekend night

The values of this variable would be

$x = 0, 1, 2, \ldots$

The distribution of X could be described as

$$X \sim Poi\,(\lambda = 2.2)$$

The probability mass function of this Poisson variable is

$$P(X = x) = \frac{2.2^x e^{-2.2}}{x!}\ , \quad x = 0, 1, 2, \ldots$$

The expected value or mean of X is

$$EX = \mu = \sum x \cdot p(x) = \lambda = 2.2$$

The variance of X is

$$\sigma^2 = \sum x^2 \cdot p(x) - \mu^2 = \lambda = 2.2$$

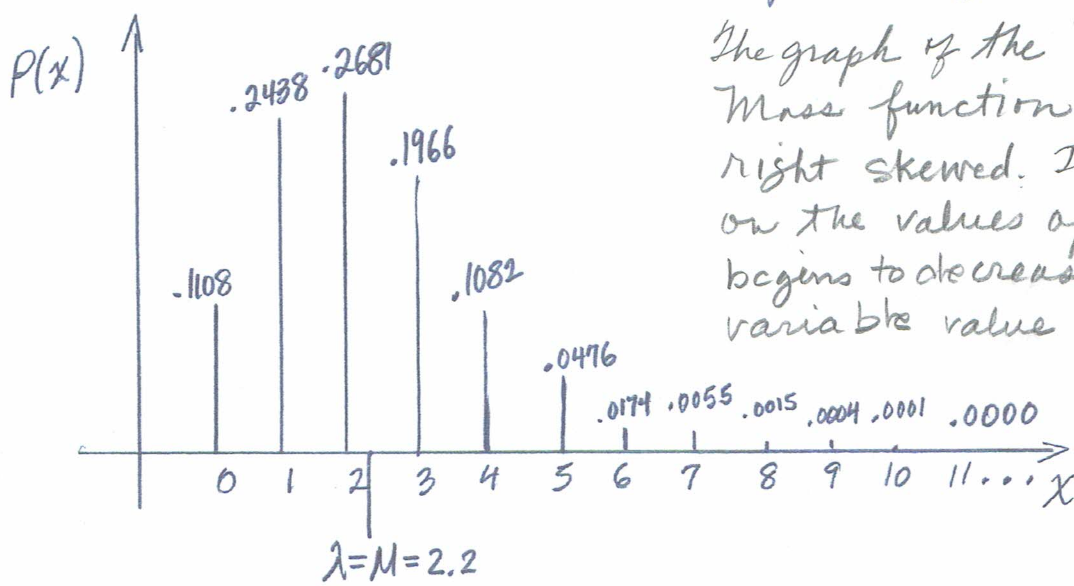
The standard deviation of X is

$$\sigma = \sqrt{\sigma^2} = \sqrt{\lambda} = \sqrt{2.2}$$

The probabilities on the values of $X \sim Poi(2.2)$ are shown in the table below

x	$P(X=x) = \dfrac{2.2^x e^{-2.2}}{x!}$
0	$P(X=0) = \dfrac{2.2^0 e^{-2.2}}{0!} = \dfrac{1\,(.110803158)}{1} = .1108$
1	$P(X=1) = \dfrac{2.2^1 e^{-2.2}}{1!} = \dfrac{2.2\,(.110803158)}{1} = .2438$
2	$P(X=2) = \dfrac{2.2^2 e^{-2.2}}{2!} = \dfrac{4.84\,(.110803158)}{2} = .2681$
3	$P(X=3) = \dfrac{2.2^3 e^{-2.2}}{3!} = \dfrac{10.648\,(.110803158)}{6} = .1966$
4	$P(X=4) = \dfrac{2.2^4 e^{-2.2}}{4!} = \dfrac{23.4256\,(.110803158)}{24} = .1082$
5	$P(X=5) = \dfrac{2.2^5 e^{-2.2}}{5!} = \dfrac{51.53632\,(.110803158)}{120} = .0476$
6	$P(X=6) = \dfrac{2.2^6 e^{-2.2}}{6!} = \dfrac{113.379904\,(.110803158)}{720} = .0174$
7	$P(X=7) = \dfrac{2.2^7 e^{-2.2}}{7!} = \dfrac{249.4357888\,(.110803158)}{5040} = .0055$
8	$P(X=8) = \dfrac{2.2^8 e^{-2.2}}{8!} = \dfrac{548.7587354\,(.110803158)}{40320} = .0015$
9	$P(X=9) = \dfrac{2.2^9 e^{-2.2}}{9!} = \dfrac{1207.269218\,(.110803158)}{362880} = .0004$
10	$P(X=10) = \dfrac{2.2^{10} e^{-2.2}}{10!} = \dfrac{2655.99279\,(.110803158)}{3628800} = .0001$

The graph of the probability mass function for a Poisson random variable with parameter value, $\lambda = 2.2$ is the following.



The graph of the Poisson Mass function is always right skewed. The probability on the values of the variable begins to decrease as soon as the variable value exceeds the mean.

Notice how the probability spikes decrease in height once the values are greater than the mean. Although very large values for this variable would only have very small probabilities, each integer to infinity would have positive probability, which implies some probability amount greater than absolute zero.

Let's consider next some probability questions for this variable.

What is the probability that this variable, $X \sim Poi(2.2)$ has values within one standard deviation of the mean?

$$P(\mu - \sigma < X < \mu + \sigma) =$$

$$= P(2.2 - \sqrt{2.2} < X < 2.2 + \sqrt{2.2}) =$$

$$= P(.71676 < X < 3.6832) =$$

Ask yourself, what values of the variable are contained in this interval?

$$= P(1 \leq X \leq 3) =$$

$$= P(X=1) + P(X=2) + P(X=3) =$$

$$= .2438 + .2681 + .1966 = .7214$$

72.14% of the probability of this variable is on values that are within one standard deviation of the mean.

What is the probability that this variable, $X \sim Poi\,(2.2)$ has values within two standard deviations of the mean?

$$P(\mu - \sigma < X < \mu + \sigma) =$$

$$= P(2.2 - 2\sqrt{2.2} < X < 2.2 + 2\sqrt{2.2}) =$$

$$= P(-.7665 < X < 5.1665) =$$

$$= P(0 \leq X \leq 5) =$$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$$

$$= .1108 + .2438 + .2681 + .1966 + .1082 + .0476$$

$$= .9751$$

97.51% of the probability of this variable is on values that are within two standard deviations of the mean.

What is the probability of more than 2 DUI's on a randomly choosen weekend night, if the mean number of DUI's per weekend night is 2.2? $X \sim Poi(2.2)$.

$$P(X > 2) = P(X \geq 3) = 1 - P(X \leq 2)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2) =$$

$$= 1 - (.1108 + .2438 + .2681) = .3773$$

What is the probability that the number of DUI's on a weekend night is at most 3 if the mean number of DUI's on a weekend night is 2.2?

$$P(X \leq 3) =$$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= .1108 + .2438 + .2681 + .1966$$

$$= .8193.$$