Lesson 9
Poisson Random Variables

Learning Objectives

Upon completion of this lesson you should be able to do the following.

1. Recognize the situation and variable description that generates a Poisson random variable.
2. Use the probability mass function to calculate probability for values of the Poisson random variable.
3. Calculate the expected value and variance for a Poisson random variable.
4. Graph the probability distribution for a Poisson random variable.
5. Calculate Poisson probabilities with the Poisson table.

Key Words

Poisson distribution, Poisson random variable, time period, spatial unit

Concepts

Different kinds of discrete probability distributions arise when specific variables are measured in certain conditions. The past lesson introduced the concept of a specific type of a discrete variable that occurs under certain conditions, the Binomial random variable. The concept in this lesson is a different discrete random variable that also occurs under a specific set of conditions. The variable that is covered in this lesson is called the Poisson random variable. What is the Poisson random variable and under what conditions does it occur?

Poisson Random Variable

The **Poisson distribution** is a probability formula that tells the probability of specific numbers of some occurrence of interest in certain independent time or spatial periods, based on the average number of occurrences . The **Poisson random variable** is the number of occurrences in a set **time period** or **spatial unit**, where the periods or units are independent. The average number of occurrences in the time period or spatial unit is assumed the same for all periods or units, that is what it means to say that the time or spatial periods are independent.

Any number of occurrences of the event of interest is possible in a set time period so the Poisson random variable has probability on the integers from zero through infinity. A very large number of occurrences in a set time period is not very likely but is possible, so the Poisson distribution is always right skewed. Notice the Poisson variable is discrete, but has infinitely many possible values. For a variable to be discrete does not imply that the variable has only a finite number of values possible. When a variable is discrete the values are referred to as countable, meaning that every value can be identified or stated. There may be infinitely many values possible with a discrete random variable but the values can be specifically stated. This is the case with the Poisson random variable.

The parameter of the Poisson probability distribution is the value of the average number of occurrences for the period or unit. The value of the average number of occurrences per unit is identified with the symbol, λ, pronounced, lamda. The value of λ identifies the specific Poisson distribution for the situation. The expected value or mean of the Poisson random variable and the variance are both equal to the value of λ. The mean and the variance of the Poisson distribution can be calculated from the general equations learned earlier for the mean and variance of a discrete random variable. For the Poisson distribution both of the equations reduce to the value λ, the value of the Poisson parameter.

To write that the variable X has a Poisson distribution with parameter λ use the symbols X ~ Poi (λ). Read as, "The variable X is distributed according to the Poisson distribution with parameter λ."

Poisson Random Variable with parameter lamda: $X \sim Poi(\lambda)$

The Variable: $X = \textit{number of occurrences in set independent time or spatial periods}$

Possible Values of the Variable: $x = 0, 1, 2, \ldots$

Poisson Probability Mass Function: $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!},\ x = 0, 1, 2, \ldots.$

Where λ *= average number of occurrences per independent unit,*

and e is the inverse natural log function.

Expected value of X : $\mu = EX = \sum xP(x) = \lambda.$

Variance of X : $\sigma^2 = E(X - \mu)^2 = \sum(x - \mu)^2 P(x) = \sum x^2 P(x) - \mu^2 = \lambda.$

Exercise with Answers

Poisson Random Variable

Assume that on average you receive 1.5 email messages per hour during the evening. You are interested in calculating the probability of various numbers of email messages per hour.

1. What is the name of the probability distribution that describes the probability of such a variable?
2. State the values of the variable.
3. Calculate the probability of each value for the variable up through the value .
4. Present the probability distribution in a table form. Include the cumulative probability in an extra column. Notice the Poisson Table has these cumulative entries for the parameter value of λ=1.5.
5. Present the probability distribution in a graph form.
6. What is the expected number of email messages for one hour during the evening?
7. What is the chance of receiving at least two email messages in one hour if on average you receive 1.5 email messages in an hour?
8. What is the probability of more than two email messages but fewer than five messages during one hour in the evening?

Answers

1. The name of the probability distribution for the variable, number of email messages in one hour, is the Poisson distribution with λ=1.5.

2. The values of the variable are $x = 0,1,2,\ldots.$

3. $P(X=0) = \frac{1.5^0 e^{-1.5}}{0!} = 0.22313016$

$P(X=1) = \frac{1.5^1 e^{-1.5}}{1!} = 0.33469524$

$P(X=2) = \frac{1.5^2 e^{-1.5}}{2!} = 0.25102143$

$P(X=3) = \frac{1.5^3 e^{-1.5}}{3!} = 0.125510715$

$P(X=4) = \frac{1.5^4 e^{-1.5}}{4!} = 0.0470665$

$P(X=5) = \frac{1.5^5 e^{-1.5}}{5!} = 0.01411995$

4.

x	P(X=x)	P(X≤x)
0	0.22313	0.22313
1	0.33470	0.55783
2	0.25102	0.80885
3	0.12551	0.93436
4	0.04707	0.98143
5	0.01412	0.99555

5. For you to do: Sketch the graph of the probability mass function for this Poisson random variable for the values 0 through 5 in the space provided below.

6. $EX = \mu = \sum xp(x) = \lambda = 1.5$

7. $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - P(X \leq 1) = 1 - 0.55783 = 0.44217.$

8. $P(2 < X < 5) = P(X \leq 4) - P(X \leq 2) = 0.98143 - 0.80885 = 0.17258.$

Exercise

Poisson Random Variable

Assume that on average you receive 0.8 email messages per hour during the evening. You are interested in calculating the probability of various numbers of email messages per hour.

1. What is the name of the probability distribution that describes the probability of such a variable?
2. State the values of the variable.
3. Calculate the probability of each value for the variable, through the value 5.
4. Present the probability distribution in a table form. Include the cumulative probability in an extra column. Notice the Poisson Table has these cumulative entries for the parameter value of $\lambda=0.8$.
5. Present the probability distribution in a graph form.
6. What is the expected number of email messages for one hour during the evening?
7. What is the chance of receiving at least two email messages in one hour if on average you receive 0.8 email messages in an hour?
8. What is the probability of more than one email messages but fewer than five messages during one hour in the evening?
9. What is the probability of receiving 2 or 3 email messages in one hour in the evening ?
10. What is the probability of receiving exactly one email message in one hour in the evening?
11. What is the probability of receiving fewer than 4 email messages in one hour during the evening?