Lesson 13
Estimation of the Population Mean

Learning Objectives

Upon completion of this lesson you should be able to do the following:

1. Describe the difference between a point estimator and an interval estimator.
2. Understand the concept of an unbiased estimator.
3. Calculate a confidence interval to estimate the population mean in the cases where the standard deviation is known.
4. Explain how the width of the interval estimate changes with changes in sample size, population standard deviation, or level of confidence.
5. Define what is meant by the phrase,"95% confidence."
6. Find the necessary sample size to build a confidence interval to estimate the population mean with stated criteria.

Key Words

population, sample, statistical inference, population parameters, sample statistics, estimation, point estimation, interval estimation, estimator, estimate, point estimate, interval estimate, confidence interval, unbiased estimator, minimum variance estimator, standard error, confidence level, error rate, bound of error

Concepts

In the field of statistics a **population** is described by observing only a subset of the population known as a **sample. Statistical inference** uses information from samples to form conclusions about population parameters. Recall that **population parameters** are unknown constants that describe the populations of variable values. **Sample statistics** are numbers calculated from samples. Sample statistics are used to **estimate** unknown parameter values. This lesson focuses on estimation of population parameter values, specifically on the estimation of the population mean.

There are two types of estimation—**point estimation** and **interval estimation**. An **estimator** is a technique or equation which, once data is applied, can be used to estimate the value of a parameter. The observed value of the estimator when the equation is applied to data is called an **estimate.** A **point estimate** is a single number calculated from a sample to estimate a population parameter. An **interval estimate** is a set or span of values with which to estimate a population parameter. A **confidence interval** is a way to construct an interval estimator so that there is a certain degree of accuracy associated with the estimator.

Point Estimation

Point estimation is a process used to generate a single number to estimate a population parameter. Three important point estimators are sample statistics that were used previously as descriptive measures.

The sample mean, $\overline{X}$, is a point estimator for the population mean, μ.

The sample variance, S^2, is a point estimator for the population variance, σ^2.

The sample standard deviation, S, is a point estimator for the population standard deviation σ.

The sample mean, $\overline{X}$, provides the best point estimator for the population mean. Point estimators are evaluated based on a variety of criteria, including bias and variance. Recall the sampling distribution of the sample mean to understand this criteria. The distribution of the sample mean represents the set of all possible sample mean values when samples of a certain size are repeatedly drawn from the same population. The following three ideas are essential. The shape of the sampling distribution of $\overline{X}$ is normal for adequately large samples. The mean of the sample average is equal to the mean of the original population from which the samples are drawn. The variance of the sample mean is the original population variance divided by the sample size. These ideas can be written in symbols in the following way.

$$\overline{X} \sim N(\mu_{\overline{X}} = \mu,\ \sigma^2_{\overline{X}} = \tfrac{\sigma^2}{n}),\ \sigma_{\overline{X}} = \tfrac{\sigma}{\sqrt{n}}.$$

A point estimator is an **unbiased estimator** for a population parameter if the expected value of the point estimator is equal to the parameter value. The sample mean is an unbiased estimator for the population mean since the expected value of the sample mean is equal to the value of the population mean.

It is desirable for an estimator to have the least possible variance. A point estimator is a **minimum variance estimator** if the point estimator has the least spread, compared to other point estimators, around the value of the parameter being estimated. The sample mean is the minimum variance estimator of the class of unbiased estimators for the population mean.

In a specific sampling situation the variance of the sample mean depends on both the spread in the original population and the sample size. The larger the sample the less the variance of the sample mean around the population mean. Recognize how important this is. The sample mean is used to estimate the population mean so the closer the sample mean values are to the population mean value the better. The variance of the sample mean determines how close the values of the sample mean are to the population mean.

Although the sample mean as a point estimator for the population mean has the desirable characteristics of unbiasedness and minimum variance it does not have the characteristic of precise accuracy. The point estimate for the population mean is not equal to the actual value of the population mean. That is, the observed sample mean is not exactly equal to the population mean. Other point estimators also lack precise accuracy. The value of S^2 is not equal to σ^2. The sample standard deviation, S, is not equal to σ, the population standard deviation that it estimates.

To understand why the observed sample mean would not be equal to the population mean recall that the sampling distribution of the sample mean is approximately normal. How much probability does a normally distributed random variable have on specific value, such as the value of the population mean? There is no probability associated with any single value for a continuously distributed random variable. The probability that the sample mean is equal to the population mean is zero.

If the sample mean is not equal to the population mean then how close is it? The typical difference between the individual sample mean values and the population mean is measured by the standard deviation of the sample mean, which is the original population standard deviation divided by the square-root of the sample size. The standard deviation of a point estimator is also called the **standard error** of a point estimator since it measures the spread of the values of the estimator around the value being estimated. The standard error of $\overline{X}$ is the same thing as the standard deviation of $\overline{X}$. Both names are correct for the measure of dispersion of the sample mean about the population mean.

Interval Estimation

Point estimators do not have precise accuracy but the standard error can be identified so the spread of the point estimator around the parameter can be gauged. The information about the spread of the estimator can be used to generate a set of values that is likely to contain the value of the parameter. The interval can be constructed so that there is reasonable confidence that the resulting numerical interval will contain the parameter being estimated.

The sampling distribution of the sample mean is approximately normal assuming an adequately large sample. This implies that the set of possible sample mean values is approximately normally distributed if the number of observations is large enough. The Empirical Rule indicates that about 68% of the sample mean values would be within one standard error of the population mean and about 95% of the sample means would be within two standard errors of the population mean. Intervals that are centered on the observed value of the sample mean and have end points that are two standard errors from the sample mean contain the value of the population mean about 95% of the time. About 5% of the time the observed sample mean value is so extreme that an interval centered on the sample mean would not contain the value of the population mean.

General Form of an Approximately 95% Interval Estimate

$$\text{Point Estimate} \pm 2\ (\text{Standard Error of the Point Estimate})$$

Confidence Intervals

A **confidence interval** is an interval estimator associated with a certain probability that the interval will contain the parameter value. To construct such an interval the researcher collects a random sample, calculates the point estimate for the parameter of interest, generates the standard error associated with the point estimate and decides a reasonable level of confidence for the situation. The **confidence level** of a confidence interval is the probability that the process will result in an interval that contains the value of the population parameter of interest. Desired confidence is usually between 90% and 99%, resulting in error rates between 10% and 1%. The **error rate,** which is denoted by α (alpha), is the chance that an extreme point estimate will be observed and that the value will center the interval so poorly that it does not contain the value of the parameter. The alpha value for a confidence interval is 1 minus the level of confidence.

Once a researcher decides on the appropriate level of confidence for the situation a number will be identified from the Z-table as the multiplier for the standard error which will result in this exact level of confidence. The multiplier for the standard error when you build a confidence interval with precise confidence is not 2, but rather a value from the Z-distribution. The value from the Z-distribution is designated by $z_{\alpha/2}$. This is a value from the standard normal or Z-distribution that has one-half the error rate in the tail area to the right of the number.

The confidence level, the variance of the original population, and the sample size all affect the width of a confidence interval. **The bound of error**, sometimes called the margin of error, for a confidence interval is the distance between the point estimate and the ends of the interval. It is the amount that is added and subtracted from point estimate to construct the interval. The width of the resulting interval is equal to two times the bound of error. The bound of error is the product of the z-multiplier times the standard error of the point estimate.

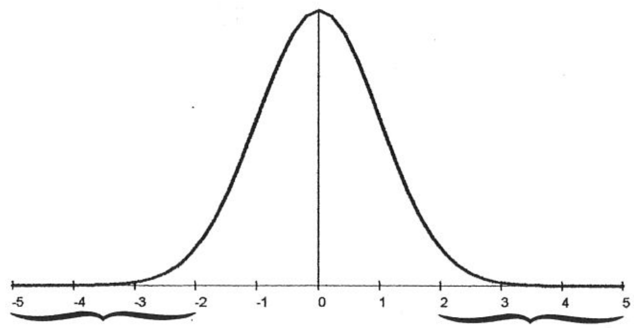
A higher confidence level for the interval results in a larger z-multiplier that increases the bound of error, thus yields a wider interval. A larger standard error also results in a larger bound of error and, thus, a wider interval. Recall the two factors that affect the magnitude of the standard error are the population variance and the sample size. A larger variance results in a larger standard error and, thus, a wider interval. A smaller sample size results in a larger standard error and, thus, a wider interval.

Confidence Interval to Estimate the Population Mean Variance known

$$\overline{X} \pm z_{\frac{\alpha}{2}} \bullet \sigma_{\overline{X}} \rightarrow \overline{X} \pm z_{\frac{\alpha}{2}} \bullet \frac{\sigma}{\sqrt{n}}$$

Interpretation of Confidence Intervals

A confidence interval provides a set of reasonable and plausible values for the parameter being estimated. But, it is not certain that the numeric confidence interval contains the parameter value. The equation used to calculate the confidence interval does not have zero error rate. Once the equation is applied to data the resulting interval either contains the parameter value or it does not. Recall that α-percent of the values of the point estimate are so extreme that the interval is centered poorly and results in an interval answer that does not contain the parameter value being estimated.



Extreme $\overline{X}$ values produce miscentered confidence intervals.

What does it mean for an interval estimate to be associated with a 95% level of confidence? Ninety-five percent of all intervals generated with such a procedure do contain the parameter of interest; five percent do not. There is a 95% chance of observing a point estimate that will center the interval adequately to result in a confidence interval that does contain the value of the parameter being estimated. There is a 5% chance of observing a point estimate that is extreme and will center the interval poorly so that the resulting interval will not contain the parameter value. A dilemma occurs when we use this process since we do not know if the observed interval generated from the randomly chosen sample is one of the 95% that really do contain the parameter of interest, or one of the 5% that do not.

Adequate Sample Size

Before a sample is drawn to produce a confidence interval the researcher must first decide how many observations to obtain. The equation below can be used to calculate the necessary number of observations to generate a confidence interval to estimate the population mean based on specific criteria. With the confidence level and the desired width of the interval set the adequate sample size to estimate the population mean can be generated if there is some way to gauge the population variance. Perhaps past research can provide information on which to base the variance. If the data is assumed mound shaped then the standard deviation can be roughly estimated by dividing the range of the data by four. State the adequate sample size as the next larger integer from the value generated by the equation given below.

$$n \geq \frac{z^2_{\frac{\alpha}{2}}\sigma^2}{B^2}.$$