Lesson 11
Normal Random Variables

Learning Objectives

Upon completion of this lesson you should be able to do the following.

1. Recognize and use the normal distribution.
2. Standardize normal random variables to be standard normal or Z random variables.
3. Calculate normal probabilities with the Z-table.
4. Calculate certain values in the normal distribution that are specific percentiles of the distribution.

Key Words

continuous, normal random variable, normal density function, standard normal random variable, standardize, standardization, Z-table,

Concepts

The last lesson introduced continuous random variables whose probability is graphically represented by area under a curve or function. The **continuous** random variables have positive probability only on intervals of values, not on specific values, and their probability structures are represented by curves. One of the most important continuous distributions in statistics is the normal distribution, which is the focus of this lesson.

Normal Random Variable

The **normal random variable** has the classic bell-shaped probability distribution. The **normal density function** is the bell curve that describes the probability associated with such a variable. The normal bell curve is perhaps the most widely recognized probability distribution and provides the basis for the approximate percentages stated in the Empirical Rule. The distributions of many naturally occurring variables, like the height of animals or trees, follow an approximate normal distribution. Averages and sums from large data sets also follow the normal distribution, which is the major reason that the normal distribution is so important to Statistics.

Assume that the random variable X has a normal distribution with a mean of μ, variance of σ^2, and a standard deviation of σ. This description implies that the probability density function would be a symmetric bell-shape curve centered at the value of the mean μ with the spread measured by the value of the standard deviation σ. The parameters of the normal distribution are the mean, μ, and variance, σ^2. For most distributions the mean and variance are functions of the parameters, but for the normal distribution the mean and variance are the parameters.

Normal Random Variable with parameters μ and σ^2 : $X \sim N(\mu, \sigma^2)$

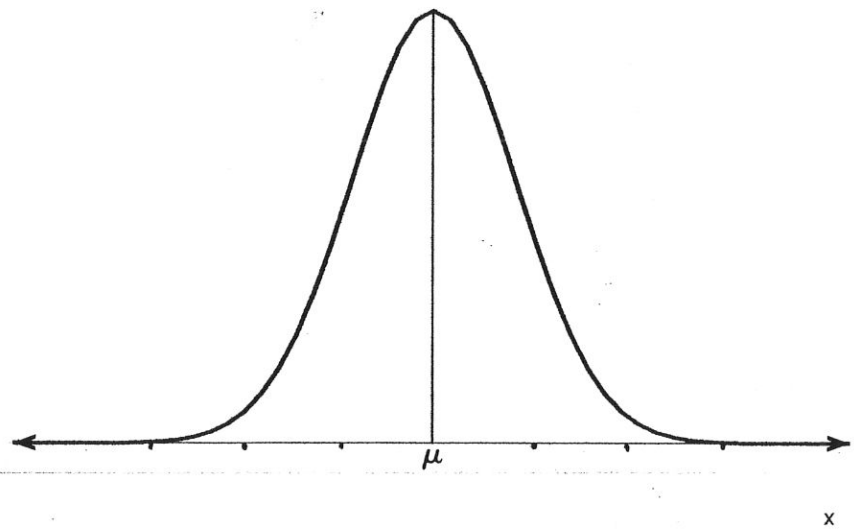
The Variable: $X = \textit{description of a normal random variable.}$

Possible Values of the Variable: $-\infty < x < \infty$.

Normal Probability Density Function: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{(x-\mu)}{\sigma}\right]^2}, -\infty < x < \infty$

Expected value of X : $\mu = EX = \mu$.

Variance of X : $\sigma^2 = E(X-\mu)^2 = \sigma^2$.



Normal Probability Density Function for $X \sim N(\mu, \sigma^2)$.

Standard Normal Random Variable

The **standard normal random variable** is a specific form of the normal random variable with a mean equal to zero and a standard deviation equal to one. The variable that has the standard normal distribution is usually called the Z variable. The Z variable is used as a reference variable for all normally distributed random variables.

This lesson discusses how to transform any normally distributed random variable to the Z-variable to provide a reference axis to calculate probabilities and generate specific percentile values. Normally distributed random variables will be changed to standard normal random variables by subtracting the mean of the variable and dividing by the standard deviation of the variable. This change mathematically alters the values of a normally distributed random variable to be values of the standard normal random variable by building z-scores out of the original variable values.

Recall your earlier work with z-scores in the section on measures of relative standing. The z-scores correspond to the values of the Z-variable if the original variable is normally distributed. Remember that z-scores tell where a value of a variable is located with respect to the mean in terms of standard deviations. The values of the reference variable Z indicate the same information about the values of a normally distributed random variable. The relative standing of a value of a normally distributed random variable is related by the z value associated with that point.

Standard Normal Random Variable with parameters 0 and 1: $Z \sim N(0,1)$

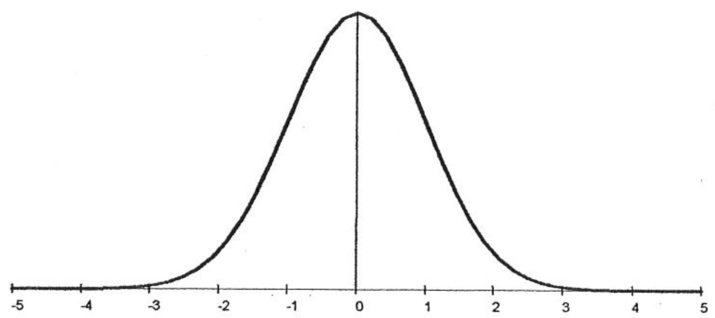
The Variable: $Z = \textit{description of the standard normal random variable.}$

Possible Values of the Variable: $-\infty < z < \infty$.

Standard Normal Probability Density Function: $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < \infty$

Expected value of Z : $\mu = EZ = 0$.

Variance of Z : $\sigma^2 = E(Z-\mu)^2 = 1$.



Standard Normal Probability Density Function for $Z \sim N(0,1)$

Standardization

The process to mathematically alter a variable with a normal distribution to be a variable with a standard normal distribution is called **standardization.** The process of standardization changes the scaling on the axis associated with the normal distribution. Standardization changes the mean of a normal distribution to be zero by substracting the mean in the numerator of the equation. Standardization changes the standard deviation of the normal distribution to be one by dividing by the original standard deviation in the equation.

To Change $X \sim N(\mu, \sigma^2)$ *to* $Z \sim N(0, 1)$:

$$If\, X \sim N(\mu, \sigma^2),\ then\ Z = [\tfrac{(X-\mu)}{\sigma}] \sim N(0, 1.).$$

To Change $Z \sim N(0, 1)$ *to* $X \sim N(\mu, \sigma^2)$:

$$If\, Z \sim N(0, 1)\ then\ X = [\mu + Z\sigma] \sim N(\mu, \sigma^2).$$

Z-Table

The Z table in your textbook gives the probability associated with an interval of z-values from zero to some positive z-value. The Z table gives the area under the bell-shaped curve for the interval from 0 to a positive z-value on the right side of the distribution. The probability associated with the interval from 0 to some positive z-value will be used to calculate the probability for any interval of interest. The probability quantities in the table can be used to generate specific percentile values for the normal distribution.

Exercise with Answers

Normal Random Variable

Assume that the monthly cost of renting a small apartment in Stillwater is normally distributed with a mean of \$380 and a standard deviation of \$55.

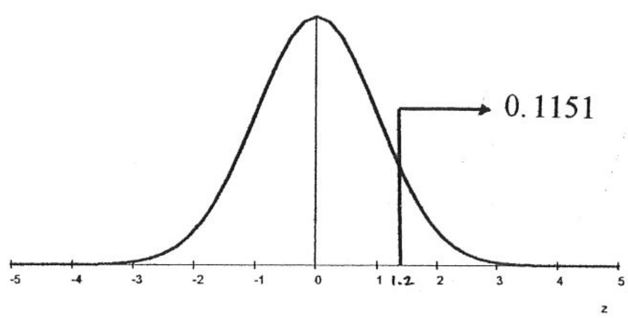
1. Identify the random variable. That is, write $X =$...and give the description of the variable.
2. State the distribution of the variable with the mean, variance and standard deviation using notation, not words.
3. What is the probability of renting a small apartment for which the rent is more than \$446? Give the probability statements for X, the transformation from X to Z, and for the Z variable. Give a graph; shade the area of interest.
4. What is the chance of renting an apartment for which the rent is between the values \$303 and \$413? Give the probability statements for X, the transformation from X to Z, and for the Z variable. Give a graph; shade the area of interest.
5. What is the probability that a randomly selected apartment rents for less than \$424? Give the probability statements for X, the transformation from X to Z, and for the Z variable. Give a graph; shade the area of interest.
6. Thirty-three percent of all the small apartments in Stillwater rent for less than what amount? Give probability statements in terms of x_o and in terms of z_o. State the equation to calculate x_o from z_o. Give a graph; shade the area of interest.
7. Of all the small apartments in Stillwater 43.7% rent for some amount between 380 and what rental cost? Give probability statements in terms of x_o and in terms of z_o. State the equation to calculate x_o from z_o. Give a graph; shade the area of interest.

Answers

1. X=monthly rent for small apartments in Stillwater

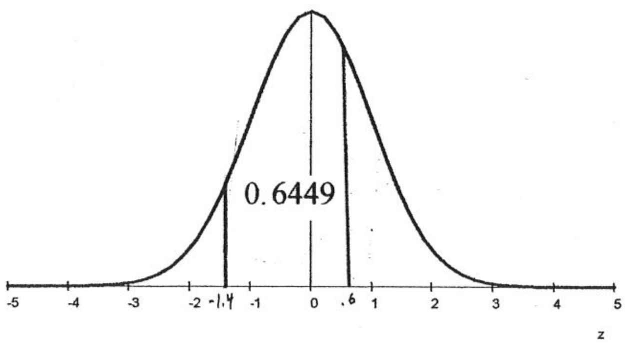
2. $X \sim N(\mu = 380, \sigma^2 = 55^2), \sigma = 55.$

3. $P(X > 446) = P\left(\frac{X-\mu}{\sigma} > \frac{446-380}{55}\right) = P(Z > 1.2) = 0.50 - 0.3849 = 0.1151.$

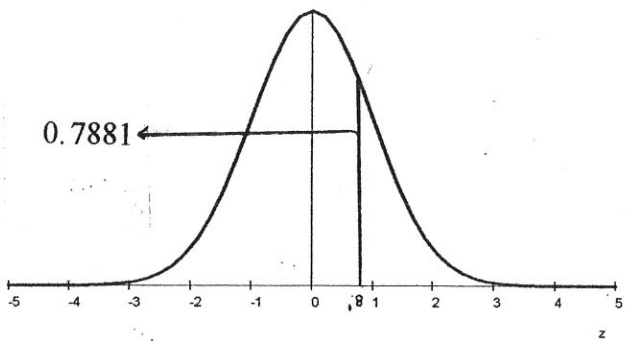


4. $P(303 < X < 413) = P\left(\frac{303-380}{55} < \frac{X-\mu}{\sigma} < \frac{413-380}{55}\right) = P(-1.4 < Z < 0.6) =$

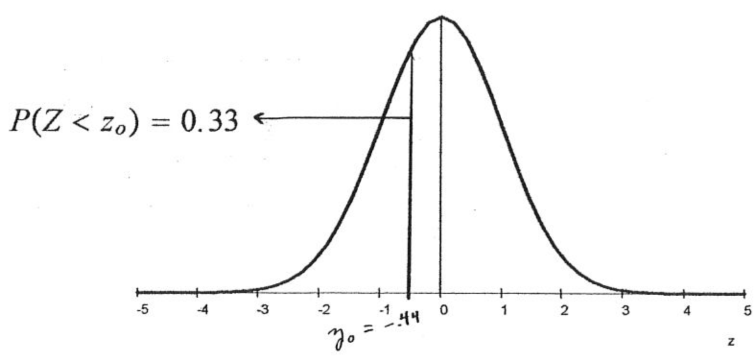
$= 0.4192 + 0.2257 = 0.6449.$



5. $P(X < 424) = P\left(\frac{X-\mu}{\sigma} < \frac{424-380}{55}\right) = P(Z < 0.8) = 0.2881 + 0.5 = 0.7881.$

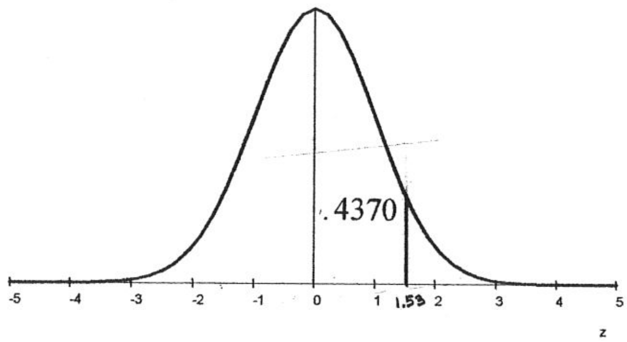


6. Find x_o, such that, $P(X < x_o) = 0.33$. First find z_o, such that, $P(Z < z_o) = 0.33$. $P(z_o < Z < 0) = 0.17$, so $z_o = -0.44$. Then $x_o = \mu + z_o\sigma = 380 + (-0.44)55 = \$355.8$.



7. Find x_o, such that, $P(380 < X < x_o) = 0.4370$. First find z_o, such that,

$P(0 < Z < z_o) = 0.4370$, so $z_o = 1.53$. Then $x_o = \mu + z_o\sigma = 380 + (1.53)55 = \$464.15$.



Exercise

Normal Random Variable

Assume that the time required to run a certain race is normally distributed with a mean of 22 minutes and a standard deviation of 1.8 minutes.

1. Identify the random variable. That is, write $X =$...and give the description of the variable.
2. State the distribution of the variable with the mean, variance and standard deviation using notation, not words.
3. What is the probability that a runner takes longer than 24.52 minutes to run the race? Give the probability statements for X, the transformation from X to Z, and for the Z variable. Give a graph; shade the area of interest.
4. What is the chance of a runner finishing the race in more than 21.1 minutes, but less than 25.6 minutes? Give the probability statements for X, the transformation from X to Z, and for the Z variable. Give a graph; shade the area of interest.
5. What is the probability that a randomly selected runner finishes the race in less than 23.35 minutes? Give the probability statements for X, the transformation from X to Z, and for the Z variable. Give a graph; shade the area of interest.
6. Thirty-three percent of all runners finish the race in less than how many minutes? Give probability statements in terms of x_o and in terms of z_o. State the equation to calculate x_o from z_o. Give a graph; shade the area of interest.
7. Of all the runners who finish the race 35.31% of them require more than 22 minutes and fewer than how many minutes to finish the race? Give probability statements in terms of x_o and in terms of z_o. State the equation to calculate x_o from z_o. Give a graph; shade the area of interest.
8. Only 2.5% of runners finish the race in less than how many minutes? Give probability statements in terms of x_o and in terms of z_o. State the equation to calculate x_o from z_o. Give a graph; shade the area of interest.
9. Five percent of runners required more than how many minutes to finish the race? Use $z_o = 1.645$. Give probability statements in terms of x_o and in terms of z_o. State the equation to calculate x_o from z_o. Give a graph; shade the area of interest.

Exercise

Normal Random Variable

Assume that the fuel mileage of a certain type of car is normally distributed with a mean of 24 miles per gallon (MPG) and a standard deviation of 1.2 MPG.

1. Identify the random variable. That is, write X=...and give the description of the variable.
2. State the distribution of the variable with the mean, variance and standard deviation using notation, not words.
3. What is the probability of this type of car having fuel mileage less than 21 MPG? Give the probability statements for X, the transformation from X to Z, and for the Z variable. Give a graph; shade the area of interest.
4. What is the chance of this type of car having fuel mileage between 21.72 and 25.56 MPG? Give the probability statements for X, the transformation from X to Z, and for the Z variable. Give a graph; shade the area of interest.
5. What is the probability that a randomly selected car of this type has fuel mileage less than 27.36? Give the probability statements for X, the transformation from X to Z, and for the Z variable. Give a graph; shade the area of interest.
6. Thirty-three percent of all the cars of this type have fuel mileage more than what amount? Give probability statements in terms of x_o and in terms of z_o. State the equation to calculate x_o from z_o. Give a graph; shade the area of interest.
7. Of all the cars of this type 20.88% have fuel mileage more than 24 MPG and less than what other value for MPG? Give probability statements in terms of x_o and in terms of z_o. State the equation to calculate x_o from z_o. Give a graph; shade the area of interest.
8. Only 2.5% of cars of this type have fuel mileage more than what amount? Give probability statements in terms of x_o and in terms of z_o. State the equation to calculate x_o from z_o. Give a graph; shade the area of interest.